Effective-medium theory for energy velocity in one-dimensional finite lossless photonic crystals

Guido Torrese,[1] Jason Taylor,[2] Trevor J. Hall,[2] and Patrice Mégret[1]
[1]*Service d'Electromagnétisme et de Télécommunications, Faculté Polytechnique de Mons, Mons, Belgium*
[2]*Centre for Research in Photonics (CRP), University of Ottawa, Ottawa, Canada*
(Received 14 October 2005; published 16 June 2006)

The effective medium theory is a useful approach for investigating the electromagnetic wave propagation in periodic multilayer slabs. It allows accurate computation of transmission and reflection spectra as well as of phase and group velocities. In this paper we derive an exact analytical expression for the energy velocity of a one-dimensional finite photonic crystal based on the effective medium approach. It accounts for the multiple reflections within the structure which results in the characteristic oscillations of the transmission spectrum. Our analytical expression holds for an arbitrary refractive index contrast and goes beyond the limits of the standard homogenization method. In order to validate our approach, results obtained by using the all-frequency effective energy velocity have been compared to those obtained using the transfer matrix method.

DOI: 10.1103/PhysRevE.73.066616 PACS number(s): 42.70.Qs, 73.40.Gk, 41.20.Jb

I. INTRODUCTION

Over the last few years photonic crystals (PCs) have received extensive attention because of the possibility of suppressing the electromagnetic wave propagation in all directions and for all polarizations across some frequency bands [1–4]. Moreover, the interesting behavior of the effective phase and group velocity near the band edge offers the possibility of designing ultrarefractive devices as well as low threshold lasers. A clear understanding of the relationship between wave transport and the band (or stop) gap is required to drive innovation. In the analysis of PC based devices the concept of phase and group velocity has been widely discussed [5,6]. Bendickson and Dowling [7] show that in the weak atom-cavity coupling limit, the group velocity can be related to the electromagnetic mode density, which in turn is proportional to the emission rate. Moreover, they provided an analytical expression for the group velocity in the case of finite, N-period, one-dimensional, linear photonic band-gap structures. Despite its apparent triviality, this is a subject which is perennially revisited whenever a novel system or experimental realization is found [8,9]. At its heart lie the physical interpretations of the various transport velocities so ably introduced by Sommerfeld and Brillouin in the early part of the previous century [10]. In contrast to the case of particles for which the velocity has a clear precise meaning, waves have at least three different kinds of velocities, i.e., the phase velocity, the group velocity and the energy velocity. The velocity of the wave packet is given by the group velocity which, in most circumstances, represents the velocity of the dominant frequency component. It is easy to see that in uniform dielectric lossless materials, because the permittivity is a slowly varying function of the angular frequency ω, phase and group velocities are equal to the velocity of energy transport, i.e., to the energy velocity [10]. On the other hand, in stratified media, for light wavelengths matching the characteristic length scale of the crystal, even when the dielectric constant is frequency independent, wave packets still experience dispersion. For finite one-dimensional photonic crystals, the phase velocity dispersion can be taken into account by means of an effective refractive index [11], while the group velocity can be calculated as a function of a complex transmission coefficient [7].

It is the purpose of this paper to discuss the energy transport mechanism in photonic crystal by introducing an analytical expression for the effective energy velocity equivalent to that provided by Bendickson and Dowling [7] for the group velocity. We strongly believe that the issues of propagation, energy flow, and field dynamics in stratified media should be properly addressed in terms of energy rather than group velocity. The work presented in this paper has been partially motivated by the recent controversy about the tunneling time associated with the velocity of evanescent waves crossing a potential barrier. As shown by Steinberg *et al.* [12], waves traveling through a photonic crystal excited within the band gap appear to be delayed less than waves propagating over the same distance in a vacuum. Similar results were obtained by Spielmann *et al.* [13], who measured the superluminal propagation of femtosecond optical pulses. The attraction of this subject comes from the apparent superluminality of wave transport through the gap, and the relation to relativistic causality. With this consideration in mind, we reexamine the concept of energy velocity, and we introduce an effective energy velocity based on an all-frequency effective medium theory.

The paper is organized as follows. Section II deals with the theory of reflection and transmission of light by stratified planar layers. We focus on simple one-dimensional translational invariant linear systems conveniently described in terms of transfer matrices. In Sec. III, we introduce the effective medium approach, while in Sec. IV we derive a simple analytical expression for the effective energy velocity. In Sec. V we present the results of our analysis.

II. SYMMETRIES IN MULTILAYER OPTICS

The dynamics of propagating waves in macroscopic stratified materials can be described by solving Maxwell's equations. The physical distinction between the fields (**E** and **H**) and the corresponding fluxes (**D** and **B**) is determined by the presence of matter through the existence of constitutive relations. These constitutive relations, which in the most gen-

eral case have a tensorial form, close the electromagnetic propagation problem by specifying the coupling of light field to the material. Since we are interested in describing a scattering process, we introduced a transfer matrix formalism. When an incident monochromatic linearly polarized plane wave travels through a stratified material, as a result of multiple reflections, forward and backward waves in the most left material are related to forward and backward waves in the most right one. When the relationship between **D** and **E**, and **B** and **H** is linear, the amplitude scaling symmetry implies the following form for the transfer matrix describing a stratified structure consisting of plane-parallel layers [14]

$$M = \begin{pmatrix} \frac{1}{T_{LR}} & -\frac{\Gamma_{RL}}{T_{LR}} \\ \frac{\Gamma_{LR}}{T_{LR}} & \frac{T_{LR}\Gamma_{RL} - \Gamma_{LR}\Gamma_{RL}}{T_{LR}} \end{pmatrix} \tag{1}$$

with Γ_{LR} and T_{LR} the wave reflection and transmission coefficients of waves traveling from left to right, respectively, while Γ_{RL} and T_{RL} refer to the corresponding coefficients for waves propagating in the opposite direction.

In general, symmetries are involved in conservation laws. Some of them are implied by the Maxwell's equations for a suitable choice of the constitutive parameters of the medium, others are related to the geometry of the structure. Provided the constitutive parameters are characterized by symmetrical tensors, a special case of which is their scalar form, the field is invariant to the interchange of source and detector. Mathematically

$$T_{LR} = T_{RL}. \tag{2}$$

Reciprocal relations of transport coefficients incorporating microscopic reversibility into a statistical mechanical treatment of irreversible linear processes have been derived by Onsager [15]. The transfer matrix of a linear reciprocal 2×2 network belongs to the unimodular group $SL(2,\mathbb{C})$ [14], regardless of the dielectric or absorbing properties of the constitutive layers. It should be noted that reciprocity does not imply energy conservation (in the form $|\Gamma|^2+|T|^2=1$), neither is it a consequence of invariance to time reversal. When a linear system is invariant under time reversal operation, i.e., there exist two indistinguishable states of the system $\{+t\}$ and $\{-t\}$, the transfer matrix M, relating the forward and backward fields in the most left material to the forward and backward fields in the most right one, can be written as

$$M = \begin{pmatrix} \frac{1}{T_{LR}} & \frac{\Gamma_{LR}^*}{T_{LR}^*} \\ \frac{\Gamma_{LR}}{T_{LR}} & \frac{1}{T_{LR}^*} \end{pmatrix}. \tag{3}$$

A scattering process invariant under time reversal satisfies

$$|\Gamma_{LR}| = |\Gamma_{RL}| \tag{4}$$

and not necessarily Eq. (2). On the other hand, systems that are reciprocal and energy conservative are invariant under time reversal operation. We now consider the case of symmetry related to the geometry of the structure, namely the translational symmetry which characterizes periodic media. The knowledge of the unit-cell transfer matrix M allows one to calculate the transmission and the reflection coefficients of the whole structure. As discussed by Bendickson and Dowling [7], a closed form expression for the transmission and the reflection coefficients of a linear time-invariant one-dimensional isotropic (and consequently reciprocal) periodic system can be obtained as

$$\frac{1}{T_N} = \frac{1}{T}\frac{\sin(N\beta(\omega))}{\sin\beta(\omega)} - \frac{\sin[(N-1)\beta(\omega)]}{\sin\beta(\omega)}, \tag{5}$$

$$\frac{\Gamma_N}{T_N} = \frac{\Gamma}{T}\frac{\sin(N\beta(\omega))}{\sin\beta(\omega)} \tag{6}$$

with T_N and Γ_N, respectively, the wave transmission and reflection coefficient of the N unit cells structure, and $\beta(\omega)$ the Bloch phase of the infinite periodic crystal obtained in the limit $N \to +\infty$. By forcing the unit cell to be invariant under coordinate inversion, the scattering process must be physically indistinguishable when looking from the right or the left. This operation which reflects the coordinates is called parity. Mathematically

$$T_{LR} = T_{RL}, \quad \Gamma_{LR} = \Gamma_{RL}. \tag{7}$$

Consequently $\Sigma_{i\neq j} m_{ij}=0$, where m_{ij} are the elements of the transfer matrix M. Parity conservation not only implies reciprocity, but also forces the left to right and right to left wave reflection coefficients to be equal. As previously discussed, when the unit cell is not invariant under parity operation, Γ_{LR} and Γ_{RL} differ only by a phase factor.

III. EFFECTIVE MEDIUM APPROACH

The effective properties associated with the propagation of electromagnetic waves in photonic crystals have been widely discussed in the literature [16,17]. The definition of effective quantities is an attempt to extend traditional concepts such as phase and group velocities, well defined in homogeneous materials, to composite materials. Although this approach can be useful for designing multilayer structures such as waveguides, it obscures the physics governing the propagation of waves. The phase velocity v_p represents the velocity of traveling equi-phase surfaces, and it is a measure of how fast different wave frequency components travel in the medium. In the absence of material dispersion in a homogeneous material, the phase velocity is a real frequency independent quantity. As discussed in [16], when considering stratified materials, the phase velocity cannot be used unambiguously. As a matter of fact, because of forward and backward plane wave superposition, the equi-phase surfaces cannot be rigorously defined. Care must be exercised when introducing an effective phase velocity, defined as the ratio of the speed of light in a vacuum c and an effective refractive index $n_{p,eff}(\omega)$ [11]. When considering frequencies near the band edge, a large change of $n_{p,eff}(\omega)$ occurs. Inside the band gap the effective refractive index is complex even in the absence of material absorption or gain. Beside the effective phase refractive index $n_{p,eff}(\omega)$, the effective group refractive

index $n_{g,eff}(\omega)$ was introduced by Sakoda [16,18] to describe the ratio $c/v_{g,eff}(\omega)$. Once more, considerations analogous to that concerning the effective phase velocity apply. Group velocity is associated with a particular wave vector and thus can only be defined for wave packets characterized by a spectrum that is narrow enough not to experience significant dispersion. Spatially confined wave packets have a spectrum containing arbitrarily high harmonics, so in order to avoid inconsistency, in the rest of this paper we restrict our analysis to signals characterized by an infinitesimal band of frequencies. Moreover, we will consider transparent media, where both k vector and the angular frequency ω are real quantities. In absorbing dielectrics, difficulties arise in attempting to relate the group velocity to the velocity of energy propagation. Because waves are attenuated as they propagate, the k vector is a complex quantity and the group velocity given by

$$v_g = \frac{\partial \omega}{\partial k} \tag{8}$$

will be complex too. On the other hand, when only the real part of the k vector is used in Eq. (8), the group velocity may be larger than c. When extending the idea of group velocity to composite materials we should introduce the term effective group velocity. It is obtained by replacing the k vector in Eq. (8) by an effective wave vector $k_{eff}(\omega)$, which in turn is related to the transmission phase $\phi(\omega)$ accumulated by a wave traveling along the medium. The effective group velocity does not have the same physical meaning as the standard group velocity. The mathematical formulation of the electromagnetic energy transport through an inhomogeneous medium in terms of effective group velocity not only implies superluminality for frequencies within the band gap, but it allows $k_{eff}(\omega)$ to be a complex quantity even in the absence of absorption [19]. The paradoxal description of superluminality arises when associating the time spent by the maximum of the field envelope crossing the medium with the dwell time. As a matter of fact, in evanescent regions, where the field decays exponentially, this conception is seriously questionable.

In order to investigate the dynamic properties of the field in photonic crystals, we consider the energy velocity v_e as defined by Brillouin [10]

$$v_e = \frac{\overline{S(\omega)}}{\overline{U(\omega)}}, \tag{9}$$

where $\overline{S(\omega)}$ is the time average magnitude of the Poynting vector and $\overline{U(\omega)}$ is the time average of the energy density. In a dielectric medium without dispersion, both permittivity and permeability are real constants, and the rate of change of the electromagnetic energy

$$U(\omega) = \frac{1}{2}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}] \tag{10}$$

has a precise thermodynamic significance: it represents the difference between the internal energy per unit volume with and without the field, the density and entropy remaining unchanged. On the other hand, when considering material dispersion, the electromagnetic energy cannot be rationally defined as a thermodynamic quantity [20].

Let us consider an electromagnetic wave propagating in the z direction in a one-dimensional medium which is electrically inhomogeneous but isotropic. Without loss of generality, the electric field is taken to be directed such that $\mathbf{E}(z,\omega)=E(z,\omega)\mathbf{x}$, and hence orthogonal to the magnetic field $\mathbf{H}(z,\omega)=H(z,\omega)\mathbf{y}$. Because of the uniformity of the medium in the x direction, the dependence on x can be taken as being through the factor $e^{jk_x x}$. The electric field inside the medium can be written as the sum of forward and backward propagating field components

$$E(z,\omega) = E^+(z,\omega) + E^-(z,\omega) \tag{11}$$

with

$$E^{\pm}(z,\omega) = E_0^{\pm}(z)e^{\mp jk(\omega)z}, \tag{12}$$

while the magnetic field is obtained from Maxwell's equations as

$$H(z,\omega) = \frac{1}{\eta}[E^+(z,\omega) - E^-(z,\omega)] \tag{13}$$

with the surface impedance $\eta=\sqrt{\mu/\epsilon}$. Time average Poynting vector and energy density can now be calculated by using Eqs. (11) and (13). By taking the real part of the complex Poynting vector, the time average of the energy flux density can be calculated as

$$\begin{aligned}\overline{S(z,\omega)} &= \frac{1}{2}\,\mathrm{Re}[E(z,\omega)\times H(z,\omega)^*] \\ &= \frac{1}{2\eta(z)}[|E^+(z,\omega)|^2 - |E^-(z,\omega)|^2],\end{aligned} \tag{14}$$

For an infinitely long structure, $|E^+(z,\omega)|^2=|E^-(z,\omega)|^2$, there are no traveling waves, the power is purely reactive and all energy is stored in the medium. On the other hand, in realistic finite structures, the forward and backward intensities are not equal, consequently there are always traveling waves transporting a real power. For the time average energy density we have

$$\begin{aligned}\overline{U(z,\omega)} &= \frac{1}{4}[\epsilon(z)|E(z,\omega)|^2 + \mu(z)|H(z,\omega)|^2] \\ &= \frac{1}{2}\epsilon(z)[|E^+(z,\omega)|^2 + |E^-(z,\omega)|^2].\end{aligned} \tag{15}$$

It represents the electromagnetic storage energy in a lossless material. When substituting Eqs. (14) and (15) into Eq. (9), the energy velocity is

$$v_e(z,\omega) = v_p(z)\frac{1-|\Gamma(z,\omega)|^2}{1+|\Gamma(z,\omega)|^2} \tag{16}$$

with $\Gamma(z,\omega)$ the wave reflection coefficient defined as

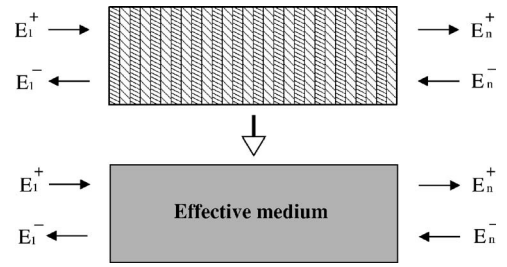

FIG. 1. Schematic of a photonic band-gap structure and the corresponding effective medium.

$$\Gamma(z,\omega) = \frac{E^-(z,\omega)}{E^+(z,\omega)}. \tag{17}$$

According to the above definition the energy velocity is a local real quantity always less than the speed of light in a vacuum. As a matter of fact, since $|\Gamma(z,\omega)|^2$ ranges from zero to one, the energy velocity is always lower than the phase velocity $v_p = c/n$, which in turn is lower than c.

Because in finite structures forward and backward traveling waves at the input and output ends of the sample determine oscillations in the reflectance, for frequency outside the band gap, the energy velocity exhibits an oscillatory behavior. On the other hand, when approaching the band-gap edges, the reflectivity increases, strongly reducing the energy velocity. When increasing the length of the structure the energy velocity at a frequency within the band gap begins to decrease, since only a small amount of energy is transmitted through the structure.

IV. EFFECTIVE ENERGY VELOCITY

In this section we describe an all frequency effective medium theory for the energy velocity. The energy transport problem is formulated in terms of electromagnetic wave propagation in a single layer with the effective properties of a photonic crystal. The effective medium approach has been widely discussed by Jeong *et al.* in [21]. The authors showed that in large frequency regions, away from the long wavelength region, a photonic crystal cannot simply be approximated as an effective medium. This result seems to contradict that obtained by Boedecker [22], who derived a simple formula for the reflection coefficient of a semi-infinite crystal described by a one-dimensional Kronig-Penney model. Because the energy velocity is directly related to the wave reflection coefficient, we first clarify this point.

Consider the transfer matrix $M_N = M^N$, characterizing a generic one-dimensional photonic band-gap structure shown in Fig. 1. This matrix is one of the possible representations of the linear map $U: \mathbb{C}^2 \to \mathbb{C}^2$, with respect to some basis. A simple factorization reducing M_N to its diagonal form is obtained as

$$M_N = (V\Pi V^{-1})^N = V\Pi^N V^{-1}, \tag{18}$$

where V is the eigenvector matrix, and $\Pi = \mathrm{diag}\{\mu_\pm\}$. Because of the transfer matrix characterizing the reciprocal N-layer structure is symplectic, the two eigenvalues are paired independently on the lattice symmetry, and they satisfy the relation $\mu_+\mu_- = 1$. From a physical point of view, the above condition implies a partition of the eigenvector into forward and backward propagating waves. Propagating modes are characterized by the condition $|\mu_\pm| = 1$, while exponentially decaying and growing modes are characterized by $|\mu_+| < 1$ and $|\mu_-| > 1$ in the forward and backward direction, respectively. By denoting forward and backward propagating waves in the Bloch space by ζ^+ and ζ^-, respectively, we have

$$\begin{pmatrix} \zeta_1^+ \\ \zeta_1^- \end{pmatrix} = \begin{pmatrix} \mu_+^N & 0 \\ 0 & \mu_-^N \end{pmatrix} \begin{pmatrix} \zeta_N^+ \\ \zeta_N^- \end{pmatrix}. \tag{19}$$

No waves are scattered back if and only if $\zeta_N^- = 0$. It should be noted that the condition $\zeta_N^- = 0$ implies that $\zeta_1^- = 0$ only if the product $\mu_-^N \zeta_N^- \to 0$ as $N \to +\infty$. While inside the band gap both forward and backward mode decay as the number of layers tends to infinity (i.e., μ_+^N, $\mu_-^{-N} \to 0$ as $N \to +\infty$), outside the band gap we have to introduce an infinitesimal amount of losses [22] to satisfy the condition $\zeta_1^- = 0$ in the limit $N \to +\infty$. We now take a closer look at Eq. (18). The diagonal matrix $\Pi^N = \mathrm{diag}\{e^{\pm j\beta(\omega)N}\}$, has exactly the structure of a propagation matrix characterized by the effective wave vector

$$k_{eff}(\omega)L = N\beta(\omega), \tag{20}$$

where L the total length of the periodic structure. On the other hand, the matrix V in the most general case does not represent an interface matrix. By noting that eigenvectors are only fixed to an overall scale factor, the most general matrix V satisfying the diagonalization problem (18) can be written as

$$V = \begin{pmatrix} v_{21}\dfrac{T\Gamma^*}{T^*(T\mu_+ - 1)} & v_{22}\dfrac{T\Gamma^*}{T^*(T\mu_- - 1)} \\ v_{21} & v_{22} \end{pmatrix}, \tag{21}$$

where the matrix elements v_{21} and v_{22} are free parameters. We can now force V to represent an interface matrix by opportunely choosing the free parameters v_{21} and v_{22}. The matrix describing the behavior of forward and backward field components at the interface is simply obtained by imposing the continuity of the tangential components of the field at the interface. By doing so, we have

$$\Delta_{sup,eff} = \frac{1}{t_{eff}} \begin{pmatrix} 1 & r_{eff} \\ r_{eff} & 1 \end{pmatrix}, \tag{22}$$

where r_{eff} and $t_{eff} = 1 + r_{eff}$ are the Fresnel reflection and transmission coefficients at the interface between the superstrate and the effective material, respectively. Thus, the matrix V represents a propagation matrix if and only if the system

$$v_{21} = v_{22}\frac{T\Gamma^*}{T^*(T\mu_- - 1)} = \frac{r_{eff}}{t_{eff}}, \tag{23}$$

$$v_{22} = v_{21}\frac{T\Gamma^*}{T^*(T\mu_+ - 1)} = \frac{1}{t_{eff}} \tag{24}$$

has a solution. After some algebra, we can show that the system above has the solution

$$v_{11} = v_{22} = \frac{\Gamma^*}{1 - T^*\mu_- + \Gamma^*}, \tag{25}$$

$$v_{12} = v_{21} = \frac{1 - T^*\mu_-}{1 - T^*\mu_- + \Gamma^*}, \tag{26}$$

if the unit-cell reflection and transmission coefficients satisfy the following equation:

$$T\Gamma^* + T^*\Gamma = 0. \tag{27}$$

For a linear system, Eq. (27) implies that the matrix M is invariant under a change of parity. It should be noted that this is not the case for the unit-cell matrix derived in [21]. This is not surprising. When Eq. (27) is not satisfied, the effective medium approach fails. In the rest of this paper we consider a unit-cell transfer matrix M invariant under parity operation.

By using the V matrix, the condition $\zeta_1^- = 0$ in the Bloch space, implies in the plane wave space

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = V\begin{pmatrix} \zeta_1^+ \\ 0 \end{pmatrix}. \tag{28}$$

Then the reflection coefficient is

$$r_{eff}(\omega) = \frac{v_{21}}{v_{11}} = \frac{\Gamma(\omega)}{1 - T(\omega)\mu_-(\omega)}. \tag{29}$$

Equation (29) represents the effective Fresnel reflection coefficient. It is a frequency dependent function of the wave reflection and transmission coefficients of the unit cell. Now the effective wave reflection coefficient $\Gamma_{eff}(\omega)$ for the slab can be calculated by multiplying the three matrices V, Π^N, and V^{-1}

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = V\Pi^N V^{-1}\begin{pmatrix} E_N^+ \\ E_N^- \end{pmatrix} \tag{30}$$

and by imposing the condition $E_N^- = 0$. The latter implies that no backward propagating waves travel in the rightmost material. By doing so the effective wave reflection coefficient can be obtained by using Eq. (17) as

$$\Gamma_{eff}(\omega) = \frac{r_{eff}(\omega)[1 - e^{-2jk_{eff}(\omega)L}]}{1 - r_{eff}(\omega)^2 e^{-2jk_{eff}(\omega)L}}, \tag{31}$$

while the effective wave transmission coefficient is

$$T_{eff}(\omega) = \frac{[1 - r_{eff}^2(\omega)]e^{-jk_{eff}(\omega)L}}{1 - r_{eff}(\omega)^2 e^{-2jk_{eff}(\omega)L}}. \tag{32}$$

The effective wave vector can be calculated by using Eq. (20), which in turn requires knowledge of the Bloch phase $\beta(\omega)$ associated with a hypothetical infinite periodic structure. As shown in [7], the cosine of the Bloch phase is a real quantity related to the wave transmission coefficient of the units cell as

$$\cos(\beta(\omega)) = \frac{1}{2}\,\mathrm{Tr}(M) = \mathrm{Re}\left\{\frac{1}{T}\right\} \tag{33}$$

with $\mathrm{Tr}(M)$ the trace of the matrix M. The Bloch phase will be real whenever $|\mathrm{Tr}(M)| \leq 2$ and complex otherwise, corresponding to the pass-band and stop-band (band-gap) conditions, respectively.

Because the effective refractive index is a frequency dependent quantity, the effective medium behaves as a dispersive material. Although Eq. (14) for the time average energy flux density remains valid even in the presence of dispersion, this is not the case for the time average energy density in Eq. (15). The latter holds only for ideal dielectrics characterized by a constant permittivity. When the medium is dispersive and transparent the time average energy density is given by [20]

$$\overline{U(z,\omega)} = \frac{1}{4}\left[\frac{\partial(\omega\epsilon_{eff}(\omega))}{\partial\omega}|E(z,\omega)|^2 + \frac{\partial(\omega\mu_{eff}(\omega))}{\partial\omega}|H(z,\omega)|^2\right]. \tag{34}$$

Equation (34) reduces to Eq. (15) when both permeability and permittivity are frequency independent. In the rest of this paper we will consider the energy velocity as seen by an observer located outside the structure. Because the leftmost and right hand materials are not dispersive, the energy density can be calculated by using Eq. (15) and consequently the energy velocity is given by Eq. (16) when replacing Γ by Γ_{eff}. Clearly, this approach does not provide insight into what is happening inside the medium.

In order to determine a closed form expression for the time average energy density, we have to know the linear dielectric response of the effective medium in terms of its permittivity. At normal incidence the Fresnel effective reflection coefficient $r_{eff}(\omega)$ is

$$r_{eff}(\omega) = \frac{\eta - \eta_{eff}(\omega)}{\eta + \eta_{eff}(\omega)}, \tag{35}$$

where η is the impedance of both the leftmost and rightmost media. In practice, thin films are constructed with refractive index profiles that are piecewise-constant functions. The whole structure is located between two semi-infinite layers of air. When limiting the analysis to index profiles formed by a series of steps, the unit-cell transfer matrix can be always written as a product of interface matrices Δ_{ij} and propagation matrices Π_{h,L_h}. The index i,j,h identifies the layer, while L_j, L_h specify the length of the layer j and h, respectively. Details concerning these matrices and their properties can be found in many textbooks. See *Principles of Optics*, for instance [23]. According to the above notation a unit-cell invariant under parity operation can be written as

$$\Pi_{i,L_i/2}\Delta_{ij}\Pi_{j,L_j}\Delta_{ji}\Pi_{i,L_i/2}. \tag{36}$$

The unit-cell transfer matrix (36) has the form (3) with wave reflection and transmission coefficients determined by using the following expressions:

$$\frac{1}{T_{LR}} = \frac{1}{t_{ij}t_{ji}}(e^{j(k_iL_i+k_jL_j)} + r_{ij}r_{ji}e^{j(k_iL_i-k_jL_j)}), \tag{37}$$

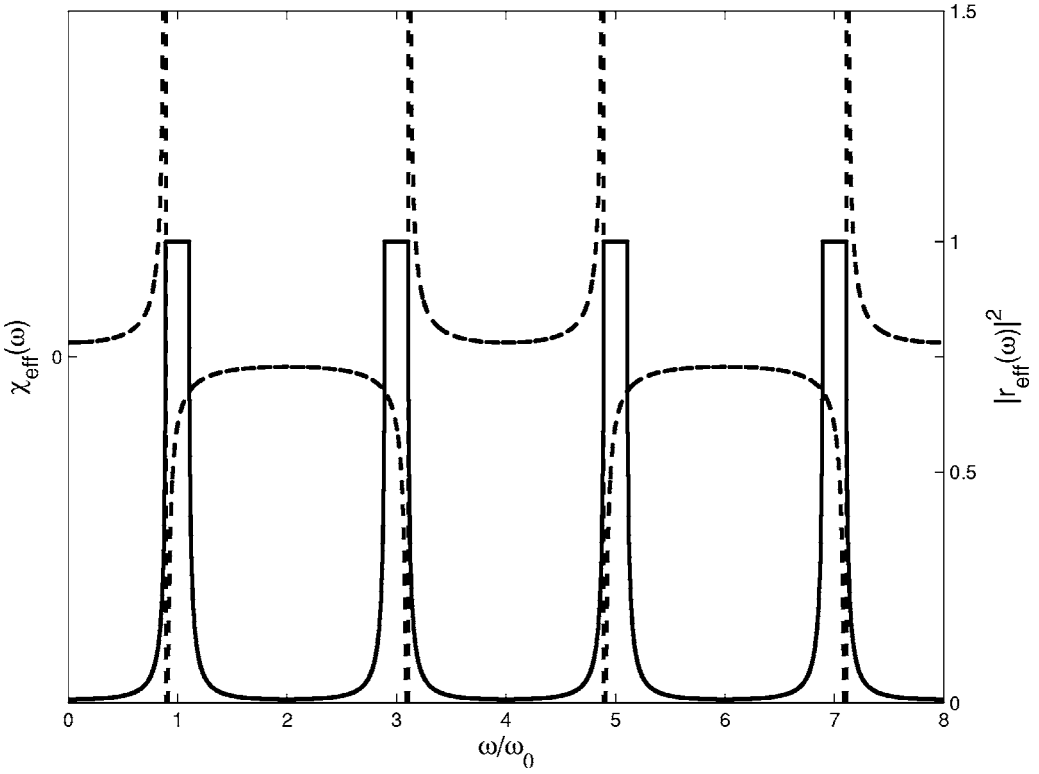

FIG. 2. Effective susceptibility and effective reflectance vs the normalized angular frequency for a $\lambda/4$ structure with $n_i=1.0$ and $n_j=1.41$.

$$\frac{\Gamma_{LR}}{T_{LR}} = \frac{1}{t_{ij}t_{ji}}(r_{ij}e^{jk_jL_j} + r_{ji}e^{-jk_jL_j}), \tag{38}$$

where r_{ij} and $r_{ji}=-r_{ij}$ are the Fresnel reflection coefficients at the ij and ji interfaces, respectively. The relative effective permittivity can now be written as

$$\epsilon_{r,eff}(\omega) = \left(\frac{\eta}{\eta_{eff}(\omega)}\right)^2 = \left(\frac{1+r_{eff}(\omega)}{1-r_{eff}(\omega)}\right)^2. \tag{39}$$

Now by using Eqs. (29), (37), and (38), with the help of (33), after a few algebraic manipulations, the effective relative permittivity can be written as

$$\begin{aligned}\epsilon_{r,eff}(\omega) &= 1 + \frac{2|\Gamma_{LR}(\omega)|^2}{\mathrm{Re}\{\Gamma_{LR}(\omega)\} - |\Gamma_{LR}(\omega)|^2} \\ &= 1 + \frac{4r_{ij}\sin(k_jL_j)}{r_{ij}(r_{ij}\sin(k_iL_i - k_jL_j) - 2\sin(k_jL_j)) - \sin(k_iL_i + k_jL_j)}\end{aligned} \tag{40}$$

with the real wave vector $k_{ij}=(\omega/c)n_{i,j}$, $L_{i,j}$ the thickness of the layers i,j used to build up the unit cell, and r_{ij} the Fresnel reflection coefficient at the interface between the medium i and j. The result given by Eq. (40) can alternatively be expressed in terms of the effective susceptibility $\chi_{eff}(\omega)$

$$\epsilon_{r,eff}(\omega) = 1 + \chi_{eff}(\omega). \tag{41}$$

The latter is always a real quantity independent of the value of the angular frequency ω. Figure 2 shows the behavior of the effective susceptibility (dashed line) versus the angular frequency, for a $\lambda/4$ structure ($L_{i,j}=\lambda_0/4n_{i,j}$) with the layer refractive indices $n_i=1.0$ and $n_j=1.41$. As for nonartificial media the effective susceptibility exhibits a set of resonances which can be described in terms of the classical Lorentz oscillator model. However, if in real crystals the multiple resonances are due to the absorption of the incident photons by bound electrons, in artificial periodic structures the Lorentzian profile is a consequence of the effective medium approach, which allows one to represent the whole structure as an equivalent material. The solid line in Fig. 2 represents the reflectance of a semi-infinite structure. It has been calculated by squaring the absolute value of Eq. (29). Everywhere the effective permittivity becomes negative, the effective refractive index switches from real to imaginary. The frequency regions characterized by a negative effective permittivity correspond to multiple stop gaps.

We can now calculate the energy velocity from the point of view of an observer located outside the structure by substituting Eq. (31) into Eq. (16), with Γ replaced by Γ_{eff}. In order to obtain a simple mathematical expression for the effective energy velocity, we discuss two separate cases. The first one corresponds to the propagation of the electromagnetic wave characterized by frequencies outside the stop band regions, while the second one deals with the evanescent nature of the field with frequencies inside the stop gaps.

A. Propagating states

Propagating states are characterized by a real Bloch phase, and by a complex conjugate pair of eigenvalues that, in the lossless case, satisfy the relation $|\mu_{\pm}|=1$. For real values of the Bloch phase, the effective permittivity is a real positive quantity and consequently $r_{eff}(\omega)$ is real. Thus, the effective energy velocity reduces to

$$v_{e,eff}(0,\omega)=c\frac{[1-r_{eff}(\omega)^2]^2}{[1+r_{eff}(\omega)^2]^2-4r_{eff}(\omega)^2\cos[2k_{eff}(\omega)L]}. \tag{42}$$

As we approach the edges of the stop gaps, the reflectance $|r_{eff}(\omega)|^2$ rises quickly to one, while the eigenvalues of the N-layer structure switch from imaginary to real. The critical values of $\beta(\omega)$ which satisfy this condition can be easily determined when considering the eigenvalues of the unit cell. They can be written as a function of the Bloch phase as

$$\mu_{\pm}=e^{\pm j\beta(\omega)}=\cos\beta(\omega)\pm j\sqrt{1-\cos^2\beta(\omega)}. \tag{43}$$

When $\cos^2\beta(\omega)=1$, the square root in Eq. (43) is zero, $\mu_{\pm}\in \mathrm{Re}$ and then $\mu_{\pm}^N\in \mathrm{Re}$. The latter implies that $\cos^2 N\beta(\omega)=1$. Thus the term $\cos[2k_{eff}(\omega)L]$ in the denominator of Eq. (42) is equal to $2\cos^2[k_{eff}(\omega)L]-1=1$. Consequently, as the frequency approaches the value corresponding to the stop-gap edges, both numerator and denominator of Eq. (42) tend to zero. In order to evaluate the effective energy velocity at the band edges we consider Eq. (31). The condition $\cos^2 N\beta(\omega)=1$ is satisfied when $|\Gamma_{eff}(\omega)|=0$, i.e., when transmission resonances occur. As a consequence at the stop-gap edges the effective energy velocity is equal to the speed of light in a vacuum.

B. Evanescent states

Evanescent modes inside the crystal correspond to eigenvectors associated with eigenvalues that are not of modulus one. If we focus our attention on the first band gap, the Bloch phase is strictly imaginary, as well as the wave vector obtained solving the dispersion relation. By consequence, the eigenvalues are real and no wave can propagate. For frequencies within the stop-gap region, the spatial z dependence of the electromagnetic field can be described as an evanescent wave, decreasing exponentially as a function of the distance from the interface. However, because of the finite length of the structure, the reflectivity within the stop gap is not exactly equal to one, so that a very small (but not zero) energy velocity characterizes the field in this frequency region. The process is very similar to that experienced by electrons crossing a barrier with a height exceeding the total particle energy. In other words, the electrodynamics of the evanescent modes is directly related to the quantum-mechanical tunneling mechanism. Within the first stop gap, the energy velocity is

$$v_{e,eff}(0,\omega)=c\frac{1-\mathrm{Re}\{r_{eff}(\omega)\}^2}{\cos[2k_{eff}(\omega)L]-\mathrm{Re}\{r_{eff}(\omega)\}^2}, \tag{44}$$

where the effective wave vector is a pure imaginary quantity. Equation (44) has been calculated taking into account the fact that $|r_{eff}(\omega)|=1$ within the stop gap. Moreover, because of the invariance of the unit-cell transfer matrix under change of parity, the real part of the refractive index (29) simplifies to

$$\mathrm{Re}\{r_{eff}(\omega)\}=\frac{\mathrm{Re}\{\Gamma(\omega)\}}{|\Gamma(\omega)|^2}. \tag{45}$$

The expressions for the effective energy velocity (42) and (44) can now be written as a function of the effective relative permittivity $\epsilon_{r,eff}(\omega)$. By inverting Eq. (39), the effective Fresnel refractive index is

$$r_{eff}(\omega)=\frac{\sqrt{\epsilon_{r,eff}(\omega)}-1}{\sqrt{\epsilon_{r,eff}(\omega)}+1}. \tag{46}$$

By substituting Eq. (46) into Eqs. (42) and (44), the following common expression for the effective energy velocity is found:

$$v_{e,eff}(0,\omega)=\frac{2\epsilon_{r,eff}(\omega)c}{[1+\epsilon_{r,eff}^2(\omega)]-[1-\epsilon_{r,eff}^2(\omega)]\cos^2[k_{eff}(\omega)L]}. \tag{47}$$

It is interesting to note that a similar expression was given by Leavens and Mayato [24] for a particle tunneling through a barrier. In that case the authors addressed the issue of determining a tunneling velocity of a point particle described by a probability distribution (i.e., wave function). While there are similarities, their work is based on different physical phenomena which are described by different wave equations. In our case we have described an effective velocity of electromagnetic energy.

V. RESULTS

Figure 3 shows the energy velocity normalized to the speed of light in a vacuum when repeating the unit cell ten times. The effective energy velocity depicted in the top panel has been computed by using Eq. (47), while the bottom panel shows a comparison between effective energy velocity and energy velocity given by Eq. (9) and calculated by applying the transfer matrix method (TMM). The two curves are in agreement within numerical error. The TMM suffers from larger numerical error near the band gaps while Eq. (47) does not.

Inside the stop band the energy velocity is very small. This situation is due to the fact that only a small amount of energy is transmitted through the structure. In the limiting case of an infinite long crystal, no electromagnetic wave can propagate, $|\Gamma_{eff}(\omega)|=1$ and the energy velocity will be zero. When this situation occurs, the interference between forward and backward carrier waves sets up a standing wave within the barrier. Because standing waves do not propagate, their velocity as well as the transmitted power is zero. On the other hand, for frequency regions outside the stop gaps, the energy velocity is always less than velocity of light in a vacuum, and it reaches its maximum at transmission resonances. This happens outside the stop gap when the unit

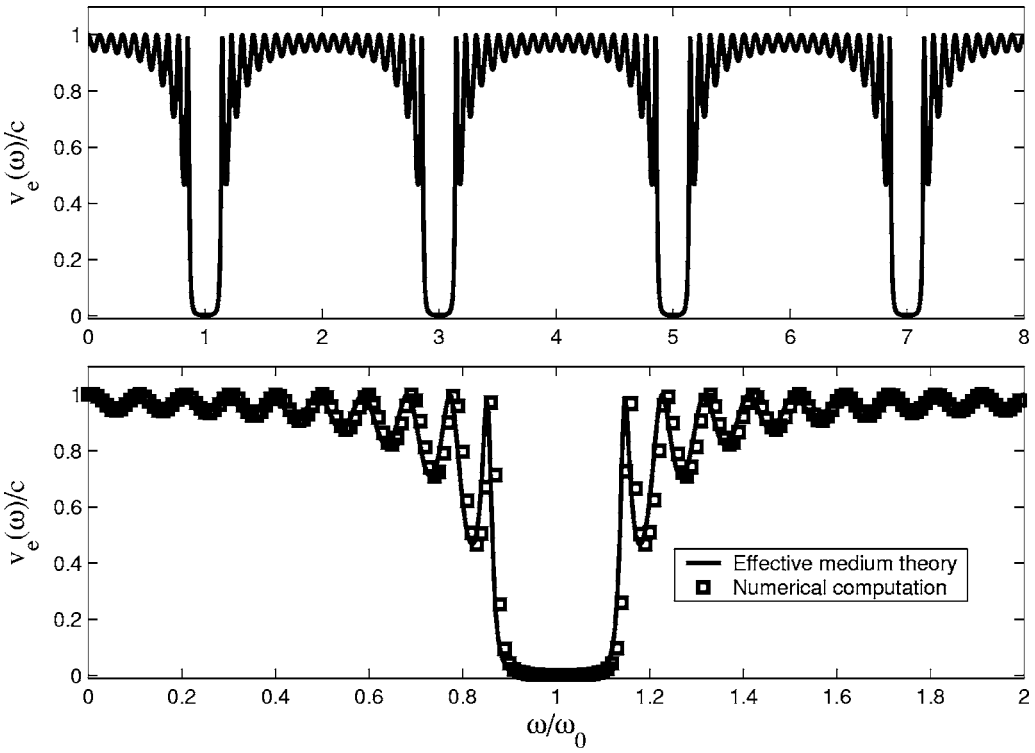

FIG. 3. Top panel: normalized effective energy velocity vs the normalized angular frequency for a ten-cell $\lambda/4$ structure with $n_i=1.0$ and $n_j=1.41$. Bottom panel: comparison between the energy velocity computed by using the effective medium approach and the energy velocity obtained by applying the transfer matrix method.

cell is transparent at some wavelength or when the angular frequency of the incident radiation satisfies the relationship

$$N\beta(\omega) = m\pi, \tag{48}$$

where m is a positive integer. As can be seen by inspecting Eq. (16), when Eq. (48) is satisfied, the energy velocity is equal to the phase velocity. Thus, the energy velocity maxima correspond to the values of the phase velocity obtained when the condition (48) holds. When the refractive index of the leftmost and rightmost materials is equal to one, the phase and energy velocities are equal to the speed of light in a vacuum. Then the energy velocity can never exceed the speed of the light in a vacuum.

VI. CONCLUSIONS

In the last few years, several research groups have shown that the group velocity can exceed the speed of light in vacuum. Although there is no problem with the possibility that the group velocity of a pulse could exceed c, there exists a fundamental difficulty with superluminal group velocities when considering information transport through the tunneling mechanism in finite structures. As a matter of fact, superluminality cannot be explained in terms of destructive interferences of traveling waves with real wave vector when only evanescent states are allowed. Fast light raises fundamental questions concerning the link between propagation delay and velocity. For a more complete and contemporary treatment of speed of information, see articles by Stenner *et al.* [25] and Sokolovski *et al.* [26].

In this paper we described the energy transport mechanism in one-dimensional finite lossless photonic crystals by utilizing the energy velocity formulation introduced by Brillouin [10]. A strictly luminal all-frequency fully analytical expression for the energy velocity has been derived in terms of effective medium theory. For evanescent modes, part of the energy is stored for a finite amount of time in the medium. The amplitude of the traveling pulse is strongly reduced because of the high wave reflection. The corresponding energy velocity is small, because the majority of the energy is stored, thus does not propagate. The effective energy velocity formulation described in this paper is a fundamental quantity to analyze the transport mechanism in finite lossless photonic crystals.

ACKNOWLEDGMENTS

The authors thank H. P. Schriemer for discussions. One of the authors also acknowledges the financial support provided by National Capital Institute of Telecommunications (NCIT).

[1] J. Lightwave Technol. **11** (1999).
[2] J. Opt. Soc. Am. B **10**, 279 (1993).
[3] In *Proceedings of the NATO ASI on Photonic Band Gap Materials*, edited by K. A. Publishers (The Netherlands, 1996).
[4] Y. Fink, J. Winn, S. Fan, C. Chen, J. Michel, J. Joannopoulos, and E. Thomas, Science **282**, 1679 (1998).
[5] J. Dowling and C. Bowden, J. Mod. Opt. **41**, 345 (1994).
[6] J. Dowling, M. Scalora, M. Bloemer, and C. Bowden, J. Appl. Phys. **75**, 1896 (1994).
[7] J. Bendickson and J. Dowling, Phys. Rev. E **53**, 4107 (1996).
[8] R. Ziolkowski, Phys. Rev. E **63**, 046604 (2001).
[9] D. Solli, C. McCormick, C. Ropers, J. Morehead, R. Chiao, and J. Hickmann, Phys. Rev. Lett. **91**, 143906 (2003).
[10] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960).
[11] M. Centini, C. Sibilia, M. Scalora, G. D'Aguanno, M. Bertolotti, M. Bloemer, C. Bowden, and I. Nefedov, Phys. Rev. E **60**, 4891 (1999).
[12] A. Steinberg, P. Kwiat, and R. Chiao, Phys. Rev. Lett. **71**, 708 (1993).
[13] C. Spielmann, R. Szipocs, A. Stingl, and F. Krauzs, Phys. Rev. Lett. **73**, 2308 (1994).
[14] J. Monźon and L. Sánchez, J. Opt. Soc. Am. A **16**, 2013 (1999).
[15] L. Onsager, Phys. Rev. **37**, 405 (1931).
[16] K. Sakoda, *Optical Properties of Photonic Crystals* (Springer, New York, 2001).
[17] S. Datta, C. Chan, K. Ho, and C. Soukoulis, Phys. Rev. B **48**, 14936 (1993).
[18] K. Sakoda, J. Opt. Soc. Am. B **14**, 1961 (1997).
[19] D. Sokolovski and L. Baskin, Phys. Rev. A **36**, 4604 (1987).
[20] L. Landau, E. Lifshitz, and L. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier Butterworth Heinemann, Linacre House, Jordan Hill, Oxford, 2002).
[21] D.-Y. Jeong, Y. Ye, and Q. Zhang, J. Appl. Phys. **92**, 4194 (2002).
[22] G. Boedecker and C. Henkel, Opt. Express **11**, 1590 (2003).
[23] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, UK, 1999).
[24] C. R. Leavens and R. S. Mayato, Ann. Phys. **7**, 662 (1998).
[25] M. D. Stenner, D. J. Gauthier, and M. A. Neifiled, Nature (London) **455**, 695 (2003).
[26] D. Sokolovski, A. Z. Msezane, and V. R. Shaginyan, Phys. Rev. A **71**, 064103 (2005).